JOURNAL OF APPROXIMATION THEORY 54, 250–259 (1988)

Berry–Esséen Bounds for (Absolute) Moments of Dominated Measures

DIETER LANDERS

Mathematisches Institut der Universität zu Köln, D-5000 Köln 41, Federal Republic of Germany

AND

LOTHAR ROGGE

Universität–GH–Duisburg, Fachbereich 11–Mathematik, D-4100 Duisburg 1, Federal Republic of Germany

Communicated by P. L. Butzer

Received April 23, 1985

1. INTRODUCTION AND NOTATION

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $1 \leqslant s \leqslant \infty$. $\mathscr{L}_s$ denotes the system of all $\mathscr{A}$-measurable $X: \Omega \to \mathbb{R}$ with $\|X\|_s < \infty$ where $\|X\|_s = (\int |X|^s \, dP)^{1/s}$ for $1 \leqslant s < \infty$ and $\|X\|_\infty = \inf\{c > 0\colon |X| \leqslant c \ P\text{-a.e.}\}$. Let $X_n \in \mathscr{L}_3$, $n \in \mathbb{N}$, be a sequence of independent and identically distributed (i.i.d.) random variables with variance $\sigma^2 > 0$. Put $S_n^* = 1/\sqrt{n}\sigma \sum_{\nu=1}^n (X_\nu - P[X_\nu])$, where $P[X_\nu] = \int X_\nu \, dP$. If $g \in \mathscr{L}_1$, denote

$$d_1(g, \sigma(X_1, ..., X_n)) := \inf\{\|g - g_0\|_1\colon g_0 \text{ is } \sigma(X_1, ..., X_n)\text{-measurable}\}.$$

Denote by Φ the standard normal distribution as well as its distribution function in $\mathbb{R}$.

In this paper we give conditions which guarantee that

$$|P[(f \circ S_n^*) \, g] - \Phi[f] \, P[g]| = O(n^{-1/2})$$

for suitable functions f and g. For $g \equiv 1$ this was one of the central problems of probability theory. Results of the above kind have been proven for $g \equiv 1$, essentially for three types of functions f, namely

(a) $f = 1_{(-\infty, t]}$, $t \in \mathbb{R}$,

(b) f is smooth and bounded,

0021-9045/88 $3.00

(c) f is smooth and fulfills certain growth conditions, e.g., $f(x)=|x|^p$ or $f(x)=x^p$.

(The "smoothness" condition in (c) has been weakened strongly in [4] to a "smoothness condition in mean.")

For general functions g there exist corresponding results for functions f of type (a) and type (b) (see [1, 2]). In this paper we give results for functions f of type (c) (see Theorem 3 and the corollaries). The methods used in this paper are different from the methods used in [1, 2]; they are more direct and seem to be more natural.

Theorem 3 of this paper yields for instance

(i) If $X_n \in \mathscr{L}_s$, $s>3$, and g is a bounded density of a p-measure Q with respect to P such that

$$d_1(g, \sigma(X_1, ..., X_n)) = O(n^{-1/2}(\lg n)^{-s/2}),$$

then

$$|Q[|S_n^*|^p] - \Phi[|x|^p]| = O(n^{-1/2})$$

for all $1 \leqslant p \leqslant s$.

(ii) If $X_n \in \mathscr{L}_s$, $s>4$, and $g \in \mathscr{L}_s$ is a density of a p-measure Q with respect to P such that

$$d_1(g, \sigma(X_1, ..., X_n)) = O(n^{-1/2}(\lg n)^{-(s-1)/2}),$$

then

$$|Q[f \circ S_n^*] - \Phi[f]| = O(n^{-1/2})$$

for each p-times differentiable f with bounded pth derivative, $p \leqslant s-1$.

2. The Results

The following concept of functions of order p is basic for this paper.

1. Definition. $f\colon \mathbb{R} \to \mathbb{R}$ is a function of order p $(p \geqslant 1)$ if

$$|f(x)-f(y)| \leqslant c\,|x-y|(1+|x|^{p-1}+|y|^{p-1}), \qquad x, y \in \mathbb{R},$$

with some suitable constant c. A function of order 1 is usually called a Lipschitz function.

The following remark gives important examples for functions of order p.

2. *Remark.* (a) If $f\colon \mathbb{R} \to \mathbb{R}$ is p-times differentiable $(p \in \mathbb{N})$ with bounded pth derivative, then f is a function of order p.

(b) If $f(x)=|x|^p$ for some $1 \leqslant p \in \mathbb{R}$ or $f(x)=x^p$ for some $1 \leqslant p \in \mathbb{N}$, then f is a function of order p.

Proof. For (a) use Taylor expansion; (b) is trivial.

The following theorem is the main result of this paper. Example 4 and the discussion below show that the assumptions of Theorem 3 are essentially optimal.

3. THEOREM. *Let $X_n \in \mathscr{L}_s$, $n \in \mathbb{N}$, be i.i.d. with positive variance. Let $g \in \mathscr{L}_r$ and $f: \mathbb{R} \to \mathbb{R}$ be a function of order p. Assume that*

$$d_1(g, \sigma(X_1, ..., X_n)) = O(n^{-1/2}(\lg n)^{-\bar{p}/2})$$

for some $\bar{p} \geqslant p$ with $\bar{p} > 3$. Then

(I) $|P[(f \circ S_n^*)\, g] - P[f \circ S_n^*]\, P[g]| = O(n^{-1/2})$ *and*

(II) $|P[(f \circ S_n^*)\, g] - \Phi[f]\, P[g]| = O(n^{-1/2})$

if $r > (s-2)/(s-3)$, $1 \leqslant p \leqslant ((r-1)/r)\, s$ for $3 < s < \infty$, or $r = \infty$, $1 \leqslant p \leqslant s$ for $s = 3$.

Proof. It suffices to prove (I). Relation (II) follows from (I), since by Theorem 1 of [4]

$$|P[f \circ S_n^*] - \Phi[f]| = O(n^{-1/2}).$$

Let w.l.o.g. $P[X_1]=0$, $P[X_1^2]=1$. There exist $\sigma(X_1, ..., X_\nu)$-measurable functions g_ν such that

$$P[|g - g_\nu|] = d_1(g, \sigma(X_1, ..., X_\nu)) \leqslant c\nu^{-1/2}(\lg \nu)^{-\bar{p}/2}. \tag{1}$$

Let $\mathbb{N}_1 = \{2^i : i \in \mathbb{N}\}$ and put

$$h_2 = g_2, \qquad h_\nu = g_\nu - g_{\nu/2} \qquad \text{for} \quad \nu \in \mathbb{N}_1, \nu \geqslant 4.$$

By (1) we have

$$P[|h_\nu|] \leqslant c\nu^{-1/2}(\lg \nu)^{-\bar{p}/2}, \qquad \nu \in \mathbb{N}_1. \tag{2}$$

Put $N_n = \{\nu \in \mathbb{N}_1 : \nu \leqslant n/2\}$ and $j(n) = \max N_n$. Then for all $n \geqslant 4$

$$g = g - g_{j(n)} + \sum_{\nu \in N_n} h_\nu.$$

Hence it suffices to prove

(A) $|P[f \circ S_n^*(g - g_{j(n)})] - P[f \circ S_n^*]\, P[g - g_{j(n)}]| = O(n^{-1/2})$,

(B) $\sum_{\nu \in N_n}(P[(f \circ S_n^*)\, h_\nu] - P[f \circ S_n^*]\, P[h_\nu]) = O(n^{-1/2})$.

Ad (A). As f is a function of order p, we have

$$f(S_n^*) = f(0) + S_n^* R(S_n^*, 0) \qquad \text{with} \qquad |R(S_n^*, 0)| \leqslant c(1 + |S_n^*|^{p-1}).$$

Hence we have to prove that

$$|P[S_n^* R(S_n^*, 0)(g - g_{j(n)})] - P[S_n^* R(S_n^*, 0)]\, P[g - g_{j(n)}]| = O(n^{-1/2}). \quad (3)$$

We have that

$$\begin{aligned} &|P[S_n^* R(S_n^*, 0)]\, P[g - g_{j(n)}]| \\ &\quad \leqslant c(P[|S_n^*| + |S_n^*|^p]\, P[|g - g_{j(n)}|]) = O(n^{-1/2}), \end{aligned} \quad (4)$$

where the last relation follows from (1) and

$$\sup_{n \in \mathbb{N}} (\|S_n^*\|_1 + \|S_n^*\|_p) < \infty;$$

Observe that $p \leqslant s$ and $j(n) \geqslant n/4$ for sufficiently large n.

Furthermore we have with $A_n = \{|S_n^*| \geqslant \sqrt{(s-1)\lg n}\}$ and $1/r' + 1/r = 1$

$$\begin{aligned} &|P[S_n^* R(S_n^*, 0)(g - g_{j(n)})]| \\ &\quad \leqslant cP[(|S_n^*| + |S_n^*|^p)\, |g - g_{j(n)}|] \\ &\quad \leqslant c(\lg n)^{p/2}\, P[|g - g_{j(n)}|] + c \int_{A_n} |S_n^*|^p\, |g - g_{j(n)}|\, dP \\ &\quad \underset{(1)}{\leqslant} O(n^{-1/2}) + cn^{-p/2} P[|S_n|^p\, 1_{A_n}\, |g - g_{j(n)}|] \end{aligned}$$

and hence by the inequality of Hölder

$$\begin{aligned} &\leqslant O(n^{-1/2}) + cn^{-p/2}\, \|\, |S_n|^p\, 1_{A_n}\|_{r'}\, \|g - g_{j(n)}\|_r \\ &\underset{(+)}{\leqslant} O(n^{-1/2}) + cn^{-p/2} n^{(p-(s-2)/r')/2}\, \|g\|_r \underset{(++)}{\leqslant} O(n^{-1/2}), \end{aligned}$$

where $(+)$ follows from (F1) (see end of the proof) and Lemma 8, and $(++)$ follows as $r \geqslant (s-2)/(s-3)$ implies $(s-2)/r' \geqslant 1$.

Together with (4) we consequently obtain (3), and hence (A).

Ad (B). Since f is a function of order p we have

$$f(S_n^*) = f\left(\frac{S_n - S_\nu}{\sqrt{n}}\right) + \frac{S_\nu}{\sqrt{n}} R\left(S_n^*, \frac{S_n - S_\nu}{\sqrt{n}}\right) \quad (5)$$

with

$$\begin{aligned} \left| R\left(S_n^*, \frac{S_n - S_\nu}{\sqrt{n}}\right)\right| &\leqslant c\left(1 + |S_n^*|^{p-1} + \left|\frac{S_n - S_\nu}{\sqrt{n}}\right|^{p-1}\right) \\ &\leqslant c + \frac{c}{n^{(p-1)/2}} (|S_\nu|^{p-1} + |S_n - S_\nu|^{p-1}). \end{aligned} \quad (6)$$

Let $\mathscr{A}_\nu = \sigma(X_1, ..., X_\nu)$ and $\nu < n$. As h_ν is $\mathscr{A}_\nu$-measurable, and hence independent from $S_n - S_\nu$, we have

$$\begin{aligned}
a_{\nu,n} &= |P[f \circ S_n^*(h_\nu - P[h_\nu])]| \\
&\underset{(5)}{=} \left| P\left[f\left(\frac{S_n - S_\nu}{\sqrt{n}}\right)(h_\nu - P[h_\nu])\right] \right. \\
&\quad \left. + P\left[\frac{S_\nu}{\sqrt{n}} R\left(S_n^*, \frac{S_n - S_\nu}{\sqrt{n}}\right)(h_\nu - P[h_\nu])\right]\right| \\
&= \frac{1}{\sqrt{n}} \left| P\left[S_\nu R\left(S_n^*, \frac{S_n - S_\nu}{\sqrt{n}}\right)(h_\nu - P[h_\nu])\right]\right| \\
&\underset{(6)}{\leqslant} \frac{c}{\sqrt{n}} P[|S_\nu|\ |h_\nu - P[h_\nu]|] \\
&\quad + c\frac{1}{n^{p/2}} P[(|S_\nu|^p + |S_\nu|\ |S_n - S_\nu|^{p-1})\ |h_\nu - P[h_\nu]|] \\
&\leqslant \frac{c}{\sqrt{n}} P[|S_\nu|\ |h_\nu|] + \frac{c}{\sqrt{n}} P[|S_\nu|]\ P[|h_\nu|] \\
&\quad + c\frac{1}{n^{p/2}} \{P[|S_\nu|^p\ |h_\nu|] + P[|S_\nu|\ |S_n - S_\nu|^{p-1}\ |h_\nu|] \\
&\quad + P[|S_\nu|^p]\ P[|h_\nu|] + P[|S_\nu|\ |S_n - S_\nu|^{p-1}]\ P[|h_\nu|]\} \\
&\leqslant c\frac{1}{n^{p/2}} P[|S_\nu|^p\ |h_\nu|] + c\frac{1}{\sqrt{n}} P[|S_\nu h_\nu|] + c\frac{1}{\sqrt{n}} \frac{1}{(\lg \nu)^{3/2}}.
\end{aligned}$$

Since $\sum_{\nu \in \mathbb{N}_1} 1/(\lg \nu)^{3/2} < \infty$, we obtain (B) from Formula (F2).

It remains to prove (F1) and (F2).

(F1) $(\int_{A_n} |S_n|^{q \cdot r'}\, dP)^{1/r'} \leqslant cn^{(q-(s-2)/r')/2}$, $1 \leqslant q \leqslant p$, where $A_n = \{|S_n^*| \geqslant \sqrt{(s-1)\lg n}\}$ and $1/r' + 1/r = 1$.

(F2) $1/n^{q/2} \sum_{\nu \in N_n} P[|S_\nu|^q\ |h_\nu|] = O(n^{-1/2})$, $1 \leqslant q \leqslant p$.

Proof of (F1). We have—using Lemma 9—where c is a general constant

$$\begin{aligned}
\int_{A_n} |S_n|^{qr'}\, dP &= ((s-1)\, n \lg n)^{(qr')/2} \int_{A_n} \left|\frac{S_n^*}{\sqrt{(s-1)\lg n}}\right|^{qr'} dP \\
&\leqslant c(n \lg n)^{(qr')/2} \sum_{k \in \mathbb{N}} P\{|S_n^*| \geqslant k^{1/(qr')} \sqrt{(s-1)\lg n}\}
\end{aligned}$$

$$\leqslant c(n \lg n)^{(qr')/2} \sum_{k \in \mathbb{N}} \left[\frac{1}{k^{2s/(qr')}} \frac{1}{(\lg n)^s} \frac{1}{n^{(s-2)/2}} \right.$$

$$\left. + 2nP\{|X_1| \geqslant ck^{1/(qr')} \sqrt{n \lg n}\} \right]$$

$$\underset{(+)}{\leqslant} cn^{(qr'-(s-2))/2} + c(n \lg n)^{qr'/2} nP\left[\left|\frac{|X_1|}{c\sqrt{n \lg n}}\right|^s\right]$$

$$\leqslant cn^{(qr'-(s-2))/2},$$

where the inequality $(+)$ follows, as $p \leqslant s(r-1)/r$ implies $s/(qr') \geqslant 1$.

Proof of (F2). The case $s=3$ and $q=1$ follows similarly as formula (15) in the proof of Theorem 2 of [1]. Let therefore $s>3$ or $q>1$. We have by Hölder

$$P[|S_\nu|^q\,|h_\nu|] \leqslant c(\nu \lg \nu)^{q/2} P[|h_\nu|] + P[|S_\nu|^q 1_{A_\nu} |h_\nu|]$$

$$\leqslant c\nu^{(q-1)/2} \delta_q(\nu) + \|h_\nu\|_r \left(\int_{A_\nu} |S_\nu|^{qr'} dP\right)^{1/r'},$$

where $\delta_1(\nu) = 1/(\lg \nu)^\gamma$ with $\gamma > 1$ and $\delta_q(\nu) = 1$ for $q > 1$. Hence (F1) and Lemma 8 imply

$$\frac{1}{n^{q/2}} \sum_{\nu \in N_n} P[|S_\nu|^q\,|h_\nu|]$$

$$\leqslant \frac{c}{n^{q/2}} \sum_{\nu \in N_n} \nu^{(q-1)/2} \delta_q(\nu) + \frac{c}{n^{q/2}} \sum_{\nu \in N_n} \nu^{(q-(s-2)/r')/2}$$

$$= O(n^{-1/2}) + \frac{c}{n^{q/2}} \sum_{\nu \in N_n} \nu^{(q-(s-2)/r')/2}. \tag{7}$$

As $(s-2)/r' \geqslant 1$, we have for $q>1$ that $\sum_{\nu \in N_n} \nu^{(q-(s-2)/r')/2} \leqslant \sum_{\nu \in N_n} \nu^{(q-1)/2} = O(n^{(q-1)/2})$. If $q=1$ and hence $s>3$ then $(s-2)/r' > 1$ and therefore $\sum_{\nu \in N_n} \nu^{(q-(s-2)/r')/2} = O(1)$. Consequently (7) implies (F2).

The preceding theorem has been proven (for $s>3$) under the three conditions

(i) $d_1(g, \sigma(X_1, ..., X_n)) = O(n^{-1/2}(\lg n)^{-\bar{p}/2})$, $\bar{p} > 3$ and $\bar{p} \geqslant p$,

(ii) $1 \leqslant p \leqslant ((r-1)/r)\,s$,

(iii) $r > (s-2)/(s-3)$.

The following discussion shows that neither condition (i) nor condition (ii) can be weakened and that in (iii) we have to asume at least $r \geqslant (s-2)/(s-3)$.

Example 4 below shows that we have to assume in (i) both $\bar{p} > 3$ and $\bar{p} \geqslant p$. Condition (ii) is "necessary" to guarantee the integrability of $(f \circ S_n^*)\, g$. Since $f \circ S_n^* \in \mathscr{L}_{s/p}$ and $g \in \mathscr{L}_r$, we have to assume that $1/(s/p) + 1/r \leqslant 1$, i.e., $1 \leqslant p \leqslant ((r-1)/r)\, s$.

A slight modification of Example 5 of [2]—with $f(x) = x$—shows that for each $r < (s-2)/(s-3)$ the approximation order $O(n^{-1/2})$ of our Theorem can be destroyed by a suitable $g \in \mathscr{L}_r$. Hence we have to assume $r \geqslant (s-2)/(s-3)$.

Similar considerations show that for the case $s = 3$ the corresponding three conditions are optimal.

Condition (i) has a different structure for the cases $p > 3$ and $p \leqslant 3$. If we assume, e.g., that

$$d_1(g, \sigma(X_1, ..., X_n)) = O(n^{-1/2}(\lg n)^{-3/2}),$$

then the proof of the preceding theorem shows that for $1 \leqslant p \leqslant 3$

$$|P[f \circ S_n^* g] - P[f \circ S_n^*]\, P[g]| = O(n^{-1/2} \lg \lg n).$$

Example 4 shows that this convergence order cannot be improved.

4. EXAMPLE. This example shows that even for i.i.d. standard normally distributed X_n, $n \in \mathbb{N}$, and bounded g, the condition

$$d_1(g, \sigma(X_1, ..., X_n)) = O(n^{-1/2}(\lg n)^{-\bar{p}/2}) \qquad (*)$$

does not imply

$$|P[f \circ S_n^* g] - P[f \circ S_n^*]\, P[g]| = |P[f \circ S_n^* g] - \Phi[f]\, P[g]| = O(n^{-1/2})$$

if $\bar{p} = 3$ or $\bar{p} < p$.

For the case $\bar{p} = 3$ we choose $f(x) = x$, for the case $\bar{p} < p$ we choose $f(x) = \operatorname{sgn}(x)\, |x|^p$. In both cases we have $\Phi[f] = 0$ and hence we have to choose a bounded g, fulfilling $(*)$, such that the sequence

$$a_n := \sqrt{n}\, P[f \circ S_n^* g], \qquad n \in \mathbb{N}$$

is unbounded.

Let $\bar{p} = 3$. Since X_n are standard normally distributed it is easy to see that there exist disjoint sets $B_\nu \in \sigma(X_1, ..., X_\nu)$ with

$$B_\nu \subset \left\{ S_\nu^* \geqslant \frac{1}{2}\sqrt{\lg \nu} \right\} \qquad \text{and} \qquad P(B_\nu) = \frac{1}{\nu^{3/2}} (\lg \nu)^{-3/2}, \qquad \nu \geqslant \nu_0.$$

Put $g = 1_B$ with $B = \sum_{\nu \geqslant \nu_0} B_\nu$. Then for $n \geqslant \nu_0$

$$d_1(g, \sigma(X_1, ..., X_n)) \leqslant \sum_{\nu > n} P(B_\nu) = \sum_{\nu > n} \frac{1}{\nu^{3/2} (\lg \nu)^{3/2}} = O(n^{-1/2}(\lg n)^{-3/2})$$

and with $f(x)=x$

$$\begin{aligned} a_n &= \sqrt{n} \sum_{\nu \geqslant \nu_0} \int_{B_\nu} S_n^* \, dP = \sum_{\nu \geqslant \nu_0} \int_{B_\nu} S_n \, dP \\ &\geqslant \sum_{\nu=\nu_0}^{n} \int_{B_\nu} S_n \, dP = \sum_{\nu=\nu_0}^{n} \int_{B_\nu} S_\nu \, dP \\ &\geqslant \frac{1}{2} \sum_{\nu=\nu_0}^{n} \sqrt{\nu \lg \nu} \, P(B_\nu) = \frac{1}{2} \sum_{\nu=\nu_0}^{n} \frac{1}{\nu \lg \nu} \geqslant c \lg \lg n. \end{aligned}$$

Let $\bar{p} < p$. Since X_n, $n \in \mathbb{N}$, are standard normally distributed there exist disjoint sets $B_\nu \in \sigma(X_1, \ldots, X_\nu)$, $\nu \in \mathbb{N}_1$, such that

$$B_\nu \subset \left\{ S_\nu^* \geqslant \frac{1}{2} \sqrt{\lg \nu} \right\} \qquad \text{and} \qquad P(B_\nu) = \frac{1}{\nu^{1/2}} (\lg \nu)^{-\bar{p}/2}, \qquad \nu_0 \leqslant \nu \in \mathbb{N}_1.$$

Put $g = 1_B$ with $B = \sum_{\nu_0 \leqslant \nu \in \mathbb{N}_1} B_\nu$. Then for $n \geqslant \nu_0$

$$d_1(g, \sigma(X_1, \ldots, X_n)) \leqslant \sum_{\mathbb{N}_1 \ni \nu > n} P(B_\nu) = O(n^{-1/2} (\lg n)^{-\bar{p}/2})$$

and with $f(x) = \operatorname{sgn}(x) \, |x|^p$ for all $n \in \mathbb{N}_1$

$$\begin{aligned} a_n &= \sqrt{n} \sum_{\nu_0 \leqslant \nu \in \mathbb{N}_1} \int_{B_\nu} \operatorname{sgn}(S_n^*) \, |S_n^*|^p \, dP \\ &\underset{(+)}{\geqslant} \sqrt{n} \int_{B_n} \operatorname{sgn}(S_n^*) \, |S_n^*|^p \, dP \geqslant c \sqrt{n} (\lg n)^{p/2} \, P(B_n) \\ &= c (\lg n)^{(p - \bar{p})/2} \xrightarrow[n \in \mathbb{N}_1]{} \infty, \end{aligned}$$

where $(+)$ follows from

$$\int_{B_\nu} \operatorname{sgn}(S_n^*) \, |S_n^*|^p \, dP \geqslant 0 \qquad \text{for all } \nu_0 \leqslant \nu \in \mathbb{N}_1$$

which can be seen by direct computation.

5. Corollary. *Let $X_n \in \mathscr{L}_s$, $n \in \mathbb{N}$, be i.i.d. with positive variance and $s > 4$. Let $g \in \mathscr{L}_s$ be a density of a p-measure Q with respect to P and assume that*

$$d_1(g, \sigma(X_1, \ldots, X_n)) = O(n^{-1/2} (\lg n)^{-(s-1)/2}).$$

Then for all $p \in \mathbb{R}$ with $1 \leqslant p \leqslant s - 1$

$$|Q[|S_n^*|^p] - \Phi[|x|^p]| = O(n^{-1/2}),$$

and for all $p \in \mathbb{N}$ with $1 \leqslant p \leqslant s-1$

$$|Q[(S_n^*)^p] - \Phi[x^p]| = O(n^{-1/2}).$$

Proof. We have $\bar{p} := s-1$ and $\bar{p} \geqslant p$. Furthermore $r := s > (s-2)/(s-3)$, and $1 \leqslant p \leqslant s(r-1)/r$ for $1 \leqslant p \leqslant s-1$. Moreover $f(x) = |x|^p$, respectively $f(x) = x^p$, are functions of order p (see Remark 2b). Hence the assertion follows from Theorem 3, using $P[g] = 1$.

6. COROLLARY. *Let $X_n \in \mathscr{L}_s$, $n \in \mathbb{N}$, be i.i.d. with positive variance and $s > 4$. Let $g \in \mathscr{L}_s$ be a density of a p-measure Q with respect to P and assume that*

$$d_1(g, \sigma(X_1, ..., X_n)) = O(n^{-1/2}(\lg n)^{-(s-1)/2}).$$

Let f be a p-times differentiable function with bounded pth derivative, where $p \leqslant s-1$. Then

$$|Q[f \circ S_n^*] - \Phi[f]| = O(n^{-1/2}).$$

Proof. Direct consequence of Theorem 3 and Remark 2a.

The next corollary is an extension of a result of [2] from bounded to arbitrary Lipschitz functions.

7. COROLLARY. *Let $X_n \in \mathscr{L}_s$, $n \in \mathbb{N}$, be i.i.d. with positive variance. Let $g \in \mathscr{L}_r$ be a density of Q with respect to P and assume that*

$$d_1(g, \mathscr{A}_n) = O(n^{-1/2}(\lg n)^{-(3/2+\varepsilon)}) \qquad \textit{for some } \varepsilon > 0.$$

Then we have for each Lipschitz function f

$$|Q[f \circ S_n^*] - \Phi[f]| = O(n^{-1/2})$$

if $r > (s-2)/(s-3)$ for $s > 3$ and $r = \infty$ for $s = 3$.

Proof. Direct consequence of Theorem 3.

For the sake of completeness we cite the following two lemmas. Lemma 8 is Lemma 5 of [2], Lemma 9 is proven in [3].

8. LEMMA. *Let $1 \leqslant r \leqslant \infty$ and $g \in \mathscr{L}_r$. Let $\mathscr{A}_0 \subset \mathscr{A}$ be a sub-σ-field and g_0 a $\mathscr{A}_0$-measurable function with $\|g - g_0\|_1 = d_1(g, \mathscr{A}_0)$. Then $\|g_0\|_r \leqslant 2\,\|g\|_r$.*

9. LEMMA. *Let $X_n \in \mathscr{L}_s$, $n \in \mathbb{N}$, $s \geqslant 3$ be i.i.d. with mean 0 and variance 1. Then there exist constants c_1 and c_2 such that for $t \geqslant \sqrt{(s-1)\lg n}$*

$$P\{|S_n^*| \geqslant t\} \leqslant c_1 \frac{1}{t^{2s} n^{(s-2)/2}} + 2nP\{|X_1| > c_2 t\sqrt{n}\}.$$

References

1. D. Landers and L. Rogge, Exact approximation orders in the conditional central limit theorem, *Z. Wahrsch. Verw. Gebiete* **66** (1984), 227–244.
2. D. Landers and L. Rogge, Uniform normal approximation orders for families of dominated measures, *J. Approx. Theory* **45** (1985), 99–121.
3. R. Michel, Nonuniform central limit bounds with applications to probabilities of deviations, *Ann. Probab.* **4** (1976), 102–106.
4. T. J. Sweeting, Speeds of convergence for the multidimensional central limit theorem, *Ann. Probab.* **5** (1977), 28–41.